

# Generalized slow-roll inflation

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## Abstract

The slow-roll approximation to inflation is ultimately justified by the presence of inflationary attractors for the orbits of the solutions of the dynamical equations in phase space. There are many indications that the inflaton field couples nonminimally to the spacetime curvature: the existence of attractor points for inflation with nonminimal coupling is demonstrated, subject to a condition on the inflaton potential and the value of the coupling constant.

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A period of inflationary expansion of the early universe has come to be widely accepted in order to solve the horizon, flatness and monopole problems that plague the standard big bang cosmology. As a bonus, inflation provides quantum fluctuations of the inflaton field as a much needed mechanism for generating density perturbations, the seeds of structures observed in the universe today [1, 2].

Most of the proposed inflationary scenarios are based on the slow-roll approximation, which assumes that the kinetic energy of the inflaton is much smaller than its potential energy, and that the cosmic expansion is nearly exponential; the slow-roll approximation allows one to predict amplitudes and spectra of scalar and tensor perturbations and the features of the Doppler peaks in the cosmic microwave background (see [2] for a recent review).

There are many indications (summarized in [3]) that the inflaton couples explicitly to the Ricci curvature of spacetime  $R$ , as described by the action

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa} - \frac{\xi \phi^2 R}{2} - \frac{1}{2} \nabla^\mu \phi \nabla_\mu \phi - V(\phi) \right] , \quad (1)$$

where  $\kappa \equiv 8\pi G$ ,  $G$  is Newton's constant,  $\phi$  the inflaton field and  $V(\phi)$  its potential. Recently, much attention has been paid to nonminimally coupled models of inflation [4, 5, 6], dark matter [7], and quintessence [8]. The slow-roll approximation that allows to solve for the dynamics of inflation requires the existence of inflationary attractors for the orbits of the solutions in phase space [9, 10]. While such attractors are well known to exist for minimal coupling ( $\xi = 0$ ), their presence is *assumed*, explicitly [11] or tacitly, for nonminimal coupling ( $\xi \neq 0$ ). Although slow-roll parameters have been introduced, and a Hubble slow-roll formalism developed, for  $\xi \neq 0$  [12, 13, 11], scenarios based on this approximation would be empty theories if inflationary attractors in phase space did not exist. In this Letter *general* potentials  $V(\phi)$  and values of  $\xi$  are considered and it is demonstrated that, subject to the condition (23), such attractors exist in the general nonminimally coupled case.

In a spatially flat Friedmann-Robertson-Walker universe with line element

$$ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2) , \quad (2)$$

the action (1) yields the equations of motion

$$6 \left[ 1 - \xi (1 - 6\xi) \kappa \phi^2 \right] \left( \dot{H} + 2H^2 \right) - \kappa (6\xi - 1) \dot{\phi}^2 - 4\kappa V + 6\kappa \xi \phi V' = 0 , \quad (3)$$

$$\frac{\kappa}{2} \dot{\phi}^2 + 6\xi \kappa H \phi \dot{\phi} - 3H^2 (1 - \kappa \xi \phi^2) + \kappa V = 0 , \quad (4)$$

$$\ddot{\phi} + 3H\dot{\phi} + \xi R\phi + V' = 0 , \quad (5)$$

where an overdot and a prime denote, respectively, differentiation with respect to the comoving time  $t$  and to  $\phi$ . The phase space of the system is two-dimensional [14, 6] and  $(H, \phi)$  is a natural set of variables to describe the dynamics. All the fixed points of the system (3)-(5) are then de Sitter solutions with constant scalar field,

$$(H, \phi) = (H_0, \phi_0) , \quad (6)$$

subject to the constraints

$$12\xi H_0^2 \phi_0 + V'_0 = 0 , \quad H_0^2 (1 - \kappa \xi \phi_0^2) = \frac{\kappa V_0}{3} \quad (7)$$

(there are only two constraints since only two equations in the set (3)-(5) are independent). For minimal coupling the well known solutions  $(H, \phi) = (\pm\sqrt{\Lambda/3}, 0)$  ( $\Lambda > 0$ ) are recovered. In order to assess the stability of the universes (6), one considers space- and time-dependent perturbations,

$$H(t, \vec{x}) = H_0 + \delta H(t, \vec{x}) , \quad \phi(t, \vec{x}) = \phi_0 + \delta\phi(t, \vec{x}) ; \quad (8)$$

unfortunately the perturbative analysis is plagued by the usual gauge-dependence problems of general relativity. In a particular gauge, one cannot be sure that the growing or decaying modes investigated are not pure gauge modes which can be removed by coordinate transformations. We proceed by using the covariant and gauge-invariant (GI) formalism of Bardeen [15], further developed in [16, 17, 18]; a recent version for generalized theories of gravity (including nonminimally coupled scalar fields) was given in [12, 19]. We shall need Bardeen's [15] GI potentials  $\Phi_H$  and  $\Phi_A$  and the Ellis-Bruni [16] variables

$$\Delta\phi(t, \vec{x}) = \delta\phi + \frac{a}{k} \dot{\phi} \left( B - \frac{a}{k} \dot{H}_T \right) , \quad \Delta R(t, \vec{x}) = \delta R + \frac{a}{k} \dot{R} \left( B - \frac{a}{k} \dot{H}_T \right) , \quad (9)$$

where  $B$  and  $H_T$  are metric perturbations and  $k$  is a scalar harmonic eigenvalue [15]. The evolution equations for the variables  $\Phi_{H,A}$  and  $\Delta\phi$  were derived in [12]:

$$\dot{\Phi}_H + \left( \frac{\xi \kappa \phi \dot{\phi}}{1 - \kappa \xi \phi^2} - H \right) \Phi_A - \frac{\kappa}{1 - \kappa \xi \phi^2} \left\{ \xi \phi \Delta \dot{\phi} + \left[ \xi \phi \left( \frac{\dot{\phi}}{\phi} - H \right) - \frac{\dot{\phi}}{2} \right] \Delta \phi \right\} = 0 , \quad (10)$$

$$\left(\frac{k}{a}\right)^2 \Phi_H + \frac{1}{1 - \kappa \xi \phi^2} \left( \frac{3\xi^2 \kappa \phi^2}{1 - \kappa \xi \phi^2} + \frac{1}{2} \right) \kappa \dot{\phi}^2 \Phi_A - \frac{1}{1 - \kappa \xi \phi^2} \left\{ \left( \frac{3\xi^2 \kappa \phi^2}{1 - \kappa \xi \phi^2} + \frac{1}{2} \right) \kappa \dot{\phi} \Delta \dot{\phi} + \left[ \left( \frac{k}{a} \right)^2 \xi \phi - \ddot{\phi} \left( \frac{3\xi^2 \kappa \phi^2}{1 - \kappa \xi \phi^2} + \frac{1}{2} \right) \right] \kappa \Delta \phi \right\} = 0 , \quad (11)$$

$$\Phi_A + \Phi_H - \frac{2\xi \kappa \phi \Delta \phi}{1 - \kappa \xi \phi^2} = 0 , \quad (12)$$

$$\begin{aligned} & \ddot{\Phi}_H + H \dot{\Phi}_H + \left( H - \frac{\xi \kappa \phi \dot{\phi}}{1 - \kappa \xi \phi^2} \right) (2\dot{\Phi}_H - \dot{\Phi}_A) - \frac{\kappa V}{1 - \kappa \xi \phi^2} \Phi_A \\ & + \frac{\kappa}{1 - \kappa \xi \phi^2} \left\{ -\xi \phi \Delta \ddot{\phi} + \left[ \frac{\dot{\phi}}{2} - 2\xi (\dot{\phi} + H\phi) \right] \Delta \dot{\phi} \right. \\ & \left. + \left[ \xi \phi \left( \kappa p - \frac{\ddot{\phi}}{\phi} - \frac{2H\dot{\phi}}{\phi} \right) - \frac{V'}{2\kappa} \right] \kappa \Delta \phi \right\} = 0 , \end{aligned} \quad (13)$$

$$\Delta \ddot{\phi} + 3H \Delta \dot{\phi} + \left( \frac{k^2}{a^2} + \xi R + V'' \right) \Delta \phi + \dot{\phi} (3\dot{\Phi}_H - \dot{\Phi}_A) + 2(V' + \xi R \phi) \Phi_A + \xi \phi \Delta R = 0 , \quad (14)$$

where

$$p = \frac{1}{1 - \kappa \xi \phi^2} \left[ \frac{\dot{\phi}^2}{2} - V - 2\xi \phi \left( \ddot{\phi} + 3H\dot{\phi} + \frac{\dot{\phi}^2}{\phi} \right) \right] . \quad (15)$$

For the background universe (6) these yield, to first order in the GI perturbations,

$$\Phi_H = \Phi_A = \frac{\xi \kappa \phi_0}{1 - \kappa \xi \phi_0^2} \Delta \phi , \quad (16)$$

$$\Delta \ddot{\phi} + 3H_0 \Delta \dot{\phi} + \left[ \frac{k^2}{a^2} + V_0'' + \frac{\xi R_0 (1 + \kappa \xi \phi_0^2) + 2V_0' \kappa \xi \phi_0}{1 - \kappa \xi \phi_0^2} \right] \Delta \phi + \xi \phi_0 \Delta R = 0 , \quad (17)$$

plus a constraint equivalent to eqs. (7); the subscript zero denotes unperturbed quantities. The GI variables  $\Delta \phi$  and  $\Delta R$  coincide with the perturbations  $\delta \phi$  and  $\delta R$  to this order. The expression

$$\Delta R = \delta R = \frac{-6\xi \kappa \phi_0 [V_0'' + 4(1 + 3\xi) H_0^2]}{1 - \xi(1 - 6\xi) \kappa \phi_0^2} \Delta \phi , \quad (18)$$

yields

$$\Delta\ddot{\phi} + 3H_0\Delta\dot{\phi} + \left(\frac{k^2}{a^2} + \alpha\right)\Delta\phi = 0, \quad (19)$$

where

$$\alpha = \frac{V_0''\phi_0(1 - \kappa\xi\phi_0^2) - V_0'(1 - 3\kappa\xi\phi_0^2)}{\phi_0[1 - \xi(1 - 6\xi)\kappa\phi_0^2]}. \quad (20)$$

Let us consider the expanding ( $H_0 > 0$ ) de Sitter solutions (6): at late times  $t \rightarrow +\infty$  one can neglect the  $(k/a)^2 \propto e^{-2H_0 t}$  term in eq. (19) and look for solutions of the form

$$\Delta\phi(t, \vec{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3\vec{l} \Delta\phi_l(t) e^{i\vec{l}\cdot\vec{x}}, \quad \Delta\phi_l(t) = \epsilon_l e^{\beta_l t}; \quad (21)$$

the constants  $\beta_l$  must satisfy an algebraic equation with roots

$$\beta_l^{(\pm)} = \frac{3H_0}{2} \left( -1 \pm \sqrt{1 - \frac{4\alpha}{9H_0^2}} \right). \quad (22)$$

Since  $Re(\beta_l^{(-)}) < 0$  and the sign of  $Re(\beta_l^{(+)})$  depends on  $\alpha$ , one concludes that the GI perturbations  $\Delta\phi$  and  $\Delta R \propto \Delta\phi$  grow without bound unless

$$V_0'' \geq f(x) \frac{V_0'}{\phi_0}, \quad (23)$$

and that there is *instability* otherwise; here

$$x \equiv \kappa\xi\phi_0^2, \quad f(x) = \frac{1 - 3x}{1 - x} < 1. \quad (24)$$

The stability condition (23) is deduced by assuming that  $0 < x < 1$  and that  $\phi_0 \neq 0$  (the case  $\phi_0 = 0$  is discussed later). If  $x > 1$  a negative effective gravitational constant  $G_{eff} \equiv G(1 - \kappa\xi\phi_0^2)^{-1}$  arises [5]; moreover, one of the slow-roll parameters (29) diverges if the solution  $\phi(t)$  crosses the critical values  $\pm\phi_1 \equiv \pm(\kappa\xi)^{-1/2}$  (for  $\xi > 0$ ) or  $\pm\phi_2 \equiv \pm[\kappa\xi(1 - 6\xi)]^{-1/2}$  (for  $0 < \xi < 1/6$ ). Under the usual assumption that  $V$  be non-negative, the Hamiltonian constraint (4) forces  $|\phi|$  to be smaller than  $\phi_2$  [5, 6]; we further assume that  $|\phi| < \phi_1$  (if  $|\phi| > \phi_1$  the direction of the inequality (23) is reversed).

In general, the stability of the fixed points (6) with  $H_0 > 0$  depends on the form of the potential  $V(\phi)$  and on the value of  $\xi$ . However, the dependence from  $\xi$  disappears and stability ensues irrespective of the value of  $\xi$  if

- i)  $V(\phi)$  has a minimum ( $V'_0 = 0$  and  $V''_0 > 0$ ) at  $\phi_0$   
ii)  $V = m^2\phi^2/2$ , or  $\Lambda/\kappa + \lambda\phi^n$  ( $\Lambda, \lambda \geq 0$ ); the latter is stable for  $n \geq 1 + f(x)$ .

In the  $\phi_0 = 0$  case not yet considered, eqs. (10)-(14) yield

$$\Phi_H = \Phi_A = 0, \quad (25)$$

$$\Delta\ddot{\phi} + 3H_0\Delta\dot{\phi} + \left(\frac{k^2}{a^2} + \alpha_1\right)\Delta\phi = 0, \quad (26)$$

where  $\Delta R = 0$  and  $\alpha_1 = V''_0 + 4\xi\kappa V_0$ ; hence there is *stability* for  $V''_0 + 4\xi\kappa V_0 \geq 0$  and instability otherwise.

Finally, we consider the contracting ( $H_0 < 0$ ) solutions (6); in this case it is convenient to use conformal time  $\eta$  (defined by  $dt = a d\eta$ ) and the auxiliary variable  $u \equiv a\Delta\phi$ . Eq. (19) becomes

$$\frac{d^2u}{d\eta^2} + [k^2 - U(\eta)]u = 0. \quad (27)$$

where

$$U(\eta) = \left(4 - \frac{\alpha_1}{H_0^2}\right)\frac{1}{\eta^2} + \frac{2}{H_0\eta^3}, \quad (28)$$

and we used the relation  $\eta = -(aH_0)^{-1}$  valid in the background (5). Formally, eq. (27) is a one-dimensional Schrödinger equation for a quantum particle of unit mass in the potential  $U(\eta)$ ; its asymptotic solutions at large  $\eta$  (i.e.  $t \rightarrow +\infty$ ) are free waves  $u \simeq e^{\pm i k\eta}$ , and  $\Delta\phi \propto H_0\eta$  diverges. The solutions (6) with  $H_0 < 0$  are *unstable*, as in the  $\xi = 0$  case.

The case  $\phi = \pm\phi_1$  not considered above corresponds to a class of solutions with constant Ricci curvature containing a de Sitter representative (6); however the latter is clearly fine-tuned and unstable with respect to perturbations  $\Delta\phi$ .

As a conclusion, subject to the condition (23), inflationary attractors exist in non-minimally, as well as in minimally, coupled inflation and the slow-roll approximation for  $\xi \neq 0$  is meaningful. The Hubble slow-roll parameters ([12]-see also [11])

$$\epsilon_1 = \frac{\dot{H}}{H^2}, \quad \epsilon_2 = \frac{\ddot{\phi}}{H\dot{\phi}}, \quad \epsilon_3 = -\frac{\xi\kappa\phi\dot{\phi}}{H\left[1 - \left(\frac{\phi}{\phi_1}\right)^2\right]}, \quad \epsilon_4 = -\frac{\xi(1 - 6\xi)\kappa\phi\dot{\phi}}{H\left[1 - \left(\frac{\phi}{\phi_2}\right)^2\right]}, \quad (29)$$

vanish *exactly* for the solutions (6) ( $\epsilon_4$  also vanishes for conformal coupling  $\xi = 1/6$ ), and  $|\epsilon_i| \ll 1$  for the solutions attracted by (6) in the phase space<sup>1</sup>.

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<sup>1</sup>The importance of the inflationary attractors is made clear by the fact that, for the *contracting*

The spectra of scalar and tensor perturbations in nonminimally coupled inflation were calculated in [13] as special cases of more general gravity theories, and the respective spectral indices are

$$n_S = 1 + 2(2\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4) \ , \qquad n_T = 2(\epsilon_1 - \epsilon_3) \qquad (30)$$

computed at the time when the perturbations cross outside the horizon (these formulas can be found in [11]); applications to specific inflationary scenarios with nonminimal coupling will be given elsewhere.

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solutions (6), the slow-roll approximation is exact (in the sense that the slow-roll parameters (29) vanish); however, this bears no relationship with the actual inflationary solutions because the contracting spaces (6) are not attractors.

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